

HOPFIAN ℓ -GROUPS, MV-ALGEBRAS AND AF C^* -ALGEBRAS

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ABSTRACT. An algebra is said to be *hopfian* if it is not isomorphic to a proper quotient of itself. We describe several classes of hopfian and of non-hopfian unital lattice-ordered abelian groups and MV-algebras. Using Elliott classification and K_0 -theory, we apply our results to other related structures, notably the Farey-Stern-Brocot AF C^* -algebra and all its primitive quotients, including the Behnke-Leptin C^* -algebras $\mathcal{A}_{k,q}$.

1. INTRODUCTION

Since the publication of [23] and [28] the literature on hopfian algebras and spaces has expanded rapidly. The survey [40] may give an idea of the applicability of the notion of hopficity and of the various methods used to prove that structures have or do not have the hopfian property.

Also the literature on the equivalent categories of MV-algebras and unital ℓ -groups has been steadily expanding over the last thirty years. See [8, 9, 10, 11, 12, 14, 21] for a selection of recent papers, and the monographs [15, 35] for a detailed account on the relationships between these structures and rational polyhedra, AF C^* -algebras, Grothendieck topoi, Riesz spaces, multisets, etc.

Remarkably enough, the literature on hopfian MV-algebras, (unital as well as non-unital) ℓ -groups and C^* -algebras is virtually nonexistent. The aim of this paper is to give a first account of the depth and multiform beauty of this theory, with its own geometric, algebraic, and topological techniques. We will apply our results to the Farey-Stern-Brocot AF C^* -algebra \mathfrak{M}_1 , [32, 7], and all its quotients, including the Behnke-Leptin C^* -algebras $\mathcal{A}_{k,q}$, [3].

We recall that a *unital ℓ -group* is an abelian group G equipped with a translation invariant lattice-order and a distinguished *archimedean* element u , called the (strong, order) *unit*. In other words, every $0 \leq x \in G$ is dominated by some integer multiple of u . A *homomorphism* of unital ℓ -groups preserves the unit as well as the group and the lattice structure.

An *MV-algebra* is an involutive abelian monoid $A = (A, 0, \neg, \oplus)$ satisfying the equations $x \oplus \neg 0 = \neg 0$ and $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$. Equivalently, by Chang completeness theorem [15, Theorem 2.5.3], A satisfies all equations satisfied by the real unit interval $[0, 1]$ equipped with the operations $\neg x = 1 - x$ and $x \oplus y = \min(1, x + y)$. Boolean algebras coincide with MV-algebras satisfying the equation $x \oplus x = x$.

In [31, Theorem 3.9] a natural equivalence Γ is established between unital ℓ -groups and MV-algebras.

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A (universal) algebra R is *residually finite* if for any $x \neq y \in R$ there is a homomorphism h of R into a finite algebra such that $h(x) \neq h(y)$. This general notion is investigated in Malcev's book [29, p. 60], with the name "finite approximability".

The following result generalizes a group-theoretic theorem due to Malcev [28]:

Theorem 1.1. [20, Theorem 1], [29, Lemma 6, p. 287] *Every finitely generated residually finite algebra R is hopfian, meaning that every homomorphism of R onto R is an automorphism.*

In Theorem 2.1 finitely generated residually finite MV-algebras will be shown to coincide with semisimple MV-algebras whose finite rank maximal ideals are dense in their maximal spectral space.

Then as a particular case of Theorem 1.1, every finitely generated semisimple MV-algebra whose finite rank maximal ideals are dense is hopfian, (Corollary 2.2). Corollary 3.1 yields the counterpart of this result for unital ℓ -groups: if (G, u) is finitely generated and semisimple, and the finite rank maximal ideals of G are dense in the maximal spectral space of G , then (G, u) is hopfian.

In Section 4 we show that several classes of MV-algebras have the hopfian property. This is the case of finitely presented MV-algebras, finitely generated projective, and in particular, finitely generated free MV-algebras. Further, finitely generated MV-algebras with finite prime spectrum are hopfian, as well as MV-algebras whose maximal spectral space is a manifold without boundary, (Corollary 3.2 and Theorems 4.1, 4.3). Theorem 5.4 provides notable examples of non-hopfian MV-algebras. Corollary 5.6 yields hopfian and non-hopfian examples of MV-algebras satisfying any two of the three conditions of being finitely generated, semisimple, and having a dense set of finite rank maximal ideals, along with the negation of the third condition.

Each MV-algebraic result has an equivalent counterpart for unital ℓ -groups, via the categorical equivalence Γ (see Corollaries 3.1, 3.3, 4.2, 5.5, 5.6).

Sections 6 and 7 are devoted to applications outside the domain of MV-algebras and unital ℓ -groups. In Corollary 6.4 finitely generated free ℓ -groups (without a distinguished unit) are shown to be hopfian. Let \mathfrak{M}_1 be the Farey-Stern-Brocot AF C^* -algebra introduced in 1988, [32], and recently rediscovered and renamed \mathfrak{A} , [7, 17]. Theorem 7.3 shows that \mathfrak{M}_1 has a separating family of finite dimensional representations, (i.e., \mathfrak{M}_1 is *residually finite dimensional*). Composing the Γ functor with Grothendieck K_0 , in Theorem 7.4 we show that \mathfrak{M}_1 is hopfian. The proof relies on Elliott classification and Bott periodicity theorem, [18]. As shown in Corollary 7.5, the hopfian property is inherited by all primitive quotients of \mathfrak{M}_1 , including the Behnke-Leptin C^* -algebras $\mathcal{A}_{k,q}$.

We refer to [15, 35] for background on MV-algebras, to [1] and [6] for unital ℓ -groups, and to [18] for AF C^* -algebras, K_0 and Elliott classification. Unless otherwise specified, all MV-algebras and all unital ℓ -groups in this paper are nontrivial. To help the reader, in Section 8 we record the most basic MV-algebraic theorems used throughout Sections 2-7.

2. CHARACTERIZING RESIDUALLY FINITE MV-ALGEBRAS AND UNITAL ℓ -GROUPS

The spectral topology of an MV-algebra, [35, §4]. Unless otherwise specified, all ideals considered in this paper are proper. For any MV-algebra A we let $\text{Spec}(A)$ denote the space of its *prime* ideals—those ideals \mathfrak{p} of A such that A/\mathfrak{p} is an MV-chain. Following [35, Definition 4.14], $\text{Spec}(A)$ comes equipped with the *spectral*, or *hull kernel* topology: its closed sets have the form $F_j = \{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{p} \supseteq j\}$, where j ranges over ideals of A , plus the trivial ideal A . The resulting topological space is known as the (*prime*) *spectral space* of A .

The (*hull kernel*) topology of the maximal spectral space $\mu(A) \subseteq \text{Spec}(A)$ coincides with the topology inherited from $\text{Spec}(A)$ by restriction. The standard basis of closed sets of $\mu(A)$ is given by all sets of the form $B_a = \{\mathfrak{m} \in \mu(A) \mid a \in \mathfrak{m}\}$, letting a range over elements of A .

An MV-algebra A is *semisimple* if the intersection of its maximal ideals is $\{0\}$. A maximal ideal \mathfrak{m} of A is said to have a *finite rank* if the quotient MV-algebra A/\mathfrak{m} is finite (whence A/\mathfrak{m} is automatically a finite MV-chain).

For any compact Hausdorff space X and MV-algebra B of continuous functions on X , the maximal ideal \mathfrak{m}_x of B is defined by $\mathfrak{m}_x = \{f \in B \mid f(x) = 0\}$.

Polyhedral topology, [38], [35, §§2-3]. Let $n = 1, 2, \dots$. A set $P \subseteq \mathbb{R}^n$ is said to be a *polyhedron* if it is a finite union of closed simplexes $S_i \subseteq \mathbb{R}^n$. P need not be convex, nor connected; the simplexes S_i need not have the same dimension. If we can choose each S_i with rational vertices then P is said to be a *rational polyhedron*.

A *McNaughton function* on $[0, 1]^n$ is a $[0, 1]$ -valued continuous function f together with (affine) linear polynomials p_1, \dots, p_u with integer coefficients such that for each $x \in [0, 1]^n$ there is $j \in \{1, \dots, u\}$ satisfying $f(x) = p_j(x)$.

As a first generalization, for any integers $n, m > 0$ and rational polyhedron $P \subseteq [0, 1]^n$, a function $f: P \rightarrow [0, 1]^m$ is said to be a *polyhedral \mathbb{Z} -map* if there are McNaughton functions f_1, \dots, f_m defined on $[0, 1]^n$ such that $f(x) = (f_1(x), \dots, f_m(x))$ for each $x \in P$.

More generally, given nonempty closed sets $X \subseteq [0, 1]^n$ and $Y \subseteq [0, 1]^m$, a map $g: X \rightarrow Y$ is called a *\mathbb{Z} -map* if there exist rational polyhedra $X \subseteq P \subseteq [0, 1]^n$ and $Y \subseteq Q \subseteq [0, 1]^m$, and a polyhedral \mathbb{Z} -map $f: P \rightarrow Q$ such that $g = f|_X$ = the restriction of f to X . We denote by $\mathcal{M}(X)$ the MV-algebra of \mathbb{Z} -maps $f: X \rightarrow [0, 1]$, and say that any such f is a *McNaughton function on X* .

For every rational point $r \in \mathbb{R}^n$, $\text{den}(r)$ denotes the least common denominator of the coordinates of r . We say that $\text{den}(r)$ is the *denominator* of r .

Theorem 2.1. *For any finitely generated MV-algebra A the following conditions are equivalent:*

- (i) *A is residually finite.*
- (ii) *A is semisimple and its maximal ideals of finite rank form a dense subset of $\mu(A)$.*

Proof. For definiteness, let us assume that A has n generators, $n = 1, 2, \dots$. If ψ is a homomorphism of A into a finite algebra F , then F is isomorphic to a finite product of finite MV-chains C_1, \dots, C_k , [15, Proposition 3.6.5]. For each $j = 1, \dots, k$ let γ_j denote the j th projection map of $C_1 \times \dots \times C_k$ onto C_j . Suppose an element $b \in A$ does not belong to $\ker(\psi)$. Then $\gamma_i(\psi(b)) \neq 0$ for at least one $i = 1, \dots, k$. We then have the following well known result: *If an element b of an MV-algebra B is sent by a homomorphism to a nonzero element of a finite MV-algebra F , then some homomorphism of B into a finite MV-chain C sends b to a nonzero element of C .*

(i) \Rightarrow (ii) Evidently, A has no infinitesimal ϵ . For otherwise, no homomorphism χ of A into a finite MV-algebra satisfies $\chi(\epsilon) \neq 0$, and A is not residually finite—a contradiction. Thus by [15, Proposition 3.6.4], A is semisimple.

We now consider the set of finite rank ideals of A , with the intent of proving its denseness in the maximal spectral space $\mu(A)$. Using Theorem 8.2(iii), for some nonempty closed subset X of $[0, 1]^n$ we first identify A with the MV-algebra $\mathcal{M}(X)$ of McNaughton functions on X .

Let $\iota: x \in X \mapsto \mathfrak{m}_x \in \mu(A)$ be the homeomorphism of X onto $\mu(A)$ defined in Theorem 8.4(iv). Then a maximal ideal $\mathfrak{m} \in \mu(A)$ has a finite rank iff the quotient

MV-algebra A/\mathfrak{m} is finite iff A/\mathfrak{m} is a finite MV-chain (because A/\mathfrak{m} is simple) iff the point $x_{\mathfrak{m}} = \iota^{-1}(\mathfrak{m}) \in X$ given by Theorem 8.4(vi) is rational.

Our original task now amounts to proving that X has a dense set of rational points. Let us say that a point of X is *irrational* if not all its coordinates are rational. Arguing by way of contradiction, let $v \in X$ be an irrational point together with an *open* rational n -simplex $\mathcal{N} \ni v$ such that no rational point lies in $\mathcal{N} \cap X$. By [35, Corollary 2.10] some McNaughton function $h: [0, 1]^n \rightarrow [0, 1]$ vanishes precisely on the (closed) rational polyhedron $[0, 1]^n \setminus \mathcal{N}$. Without loss of generality we may assume that the value $h(v)$ is an irrational number. The restriction $k = h|_X$ of h to X belongs to $A = \mathcal{M}(X)$.

Let ρ be a homomorphism of A onto a finite MV-chain. By Theorem 8.4, for some rational point $z \in X$ the quotient map $f \in A \mapsto f/\mathfrak{m}_z$ coincides with ρ . By definition of \mathcal{N} , the point z lies in $X \setminus \mathcal{N}$, whence $\rho(k) = k/\mathfrak{m}_z = k(z) = 0$. Thus for no homomorphism ρ of A onto a finite MV-chain we can have $\rho(k) \neq 0$. By the remark preceding the proof of (i) \Rightarrow (ii), for no homomorphism ψ of A onto a finite MV-algebra we can have $\psi(k) \neq 0$. Since $0 \neq k(v) = h(v) \in [0, 1] \setminus \mathbb{Q}$, this contradicts the assumption that A is residually finite. Thus X has a dense set of rational points, and the finite rank maximal ideals of A are dense in $\mu(A)$, as desired.

(ii) \Rightarrow (i) By Theorem 8.2(iii) we can identify A with $\mathcal{M}(X)$ for some nonempty closed space $X \subseteq [0, 1]^n$ homeomorphic to the maximal spectral space $\mu(A)$. For any nonzero $a \in A$ we will exhibit a homomorphism σ of A into a finite MV-algebra, such that $\sigma(a) \neq 0$. Let $y \in X$ satisfy $a(y) > 0$. By definition, a is the restriction to X of some McNaughton function $f: [0, 1]^n \rightarrow [0, 1]$. Since f is continuous, for some open neighborhood \mathcal{R} of y in $[0, 1]^n$, f never vanishes over \mathcal{R} . With the notational stipulations following Theorem 8.4, the assumed density in $\mu(A)$ of the set of finite rank maximal ideals yields a rational point $r \in \mathcal{R} \cap X$ such that $f(r) = a(r) > 0$. Its corresponding maximal ideal \mathfrak{m}_r determines the homomorphism

$$\sigma: l \in A = \mathcal{M}(X) \mapsto l/\mathfrak{m}_r = l(r) \in [0, 1].$$

The range of σ is the set $V = \{l(r) \mid l \in \mathcal{M}(X)\}$. Let d be the least common denominator of the coordinates of r . Since the linear pieces of every $l \in \mathcal{M}(X)$ are (affine) linear polynomials with integer coefficients, V is an MV-subalgebra of the finite MV-chain $C = \{0, 1/d, 2/d, \dots, (d-1)/d, 1\}$. (Actually, by McNaughton theorem, [15, 9.1.5], $V = C$.) Thus $\sigma(a) = a(r)$ is a nonzero member of the finite MV-algebra $V = \text{range}(\sigma)$. In conclusion, A is residually finite. \square

From Theorems 1.1-2.1 we have:

Corollary 2.2. *Every finitely generated semisimple MV-algebra A with a dense set of finite rank maximal ideals is hopfian.*

3. HOPFIAN MV-ALGEBRAS AND UNITAL ℓ -GROUPS

The spectral topology of a unital ℓ -group. An *ideal* of a unital ℓ -group is the kernel of a homomorphism of (G, u) . (In [6] ideals are called “ ℓ -ideals”.) We let $\text{Spec}(G, u)$ denote the set of *prime* ideals of (G, u) , those ideals \mathfrak{p} such that the quotient $(G, u)/\mathfrak{p}$ is totally ordered. Unless otherwise specified, all ideals of (G, u) in this paper will be *proper*, i.e., different from G . $\text{Spec}(G, u)$ comes equipped with the (*hull kernel*) *spectral topology*, whose closed sets have the form $F_j = \{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{p} \supseteq j\}$, letting j range over all ideals of (G, u) , plus the improper ideal G .

We let $\mu(G, u)$ denote the set of maximal ideals of (G, u) equipped with the topology inherited from $\text{Spec}(G, u)$ by restriction. A basis of closed sets for the *maximal spectral space* $\mu(G, u)$ is given by the family of sets $B_a = \{\mathfrak{m} \in \mu(G, u) \mid$

$a \in \mathfrak{m}$ letting a range over elements of G . Owing to the existence of a unit in G , $\mu(G, u)$ is a nonempty compact Hausdorff space, [6, 10.2.2]. From Theorem 8.5 one easily obtains a homeomorphism of $\mu(G, u)$ onto $\mu(\Gamma(G, u))$.

A maximal ideal \mathfrak{m} of (G, u) is said to have a *finite rank* if the quotient G/\mathfrak{m} is isomorphic to the additive group \mathbb{Z} with its natural order and the unit coinciding with some integer $v > 0$. Equivalently, $\mathfrak{m} \cap [0, 1]$ is a finite rank maximal ideal of the MV-algebra $\Gamma(G, u)$, (see Theorem 8.5).

We say that (G, u) is *semisimple* if the intersection of its maximal ideals is $\{0\}$. Equivalently, the MV-algebra $\Gamma(G, u)$ is semisimple. As is well known, (G, u) is semisimple iff it is archimedean iff it is isomorphic to a lattice ordered abelian group of real valued functions over the compact Hausdorff space of its maximal ideals, with the constant function 1 as the unit, [6, Corollaire 13.2.4].

Evidently, there is no residually finite unital ℓ -group (G, u) (except the trivial one, where $0 = u$). So Theorem 1.1 has no direct applicability to (G, u) . However, combining Theorem 8.5 with Theorems 1.1-2.1 we have:

Corollary 3.1. *Let (G, u) be a finitely generated semisimple unital ℓ -group (G, u) whose maximal ideals of finite rank are dense in the maximal spectral space $\mu(G, u)$. (G, u) is hopfian.*

An MV-algebra F is said to be *projective* if whenever $\psi: A \rightarrow B$ is a surjective homomorphism and $\phi: F \rightarrow B$ is a homomorphism, there is a homomorphism $\theta: F \rightarrow A$ such that $\phi = \psi \circ \theta$.

Corollary 3.2. *Any finitely presented, any finitely generated projective, and in particular any finitely generated free MV-algebra is hopfian.*

Proof. By [35, Theorem 6.3] we may identify the finitely presented MV-algebra A with $\mathcal{M}(P)$ for some rational polyhedron $P \subseteq [0, 1]^n$. It follows that A is semisimple and the rational points are dense in P . By Theorem 8.4, the finite rank maximal ideals of A are dense in $\mu(A)$. By Theorem 2.1, A is residually finite. Since A is finitely generated, then by Corollary 2.2, A is hopfian. By [35, Proposition 17.5], finitely generated projective MV-algebras are finitely presented. Finitely generated free MV-algebras are particular cases of finitely presented MV-algebras. \square

A unital ℓ -group (G, u) is said to be *projective* if whenever $\psi: (K, w) \rightarrow (H, v)$ is a surjective homomorphism and $\phi: (G, u) \rightarrow (H, v)$ is a homomorphism, there is a homomorphism $\theta: (G, u) \rightarrow (K, w)$ such that $\phi = \psi \circ \theta$.

Corollary 3.3. *Any finitely presented, as well as any finitely generated projective unital ℓ -group (G, u) is hopfian.*

Proof. We first remark that “finitely presented unital ℓ -groups” as defined categorically following Gabriel–Ulmer ([26, 19, §§II.2, IX.1]) coincide with the Γ correspondents of finitely presented MV-algebras, ([10, Theorem 2.2], [14, Remark 5.10(a)]). By Theorem 8.5, finitely generated projective unital ℓ -groups correspond via Γ to finitely generated projective MV-algebras.

The desired result then follows from Corollary 3.2, in view of the preservation properties of Γ , (Theorem 8.5). \square

The free n -generator unital ℓ -groups $(M_n, 1)$, $n = 1, 2, \dots$ For later use in this paper we introduce here the correspondents via Γ of finitely generated free MV-algebras. To this purpose we denote by $(M_n, 1)$ the unital ℓ -group of all piecewise linear continuous functions $l: [0, 1]^n \rightarrow \mathbb{R}$ such that each linear piece of l has integer coefficients: the number of pieces of l is always finite; the adjective “linear” is

understood in the affine sense; $(M_n, 1)$ is equipped with the distinguished unit given by the constant function 1 over $[0, 1]^n$. As an alternative equivalent definition of $(M_n, 1)$,

$$\Gamma(M_n, 1) = \mathcal{M}([0, 1]^n). \quad (1)$$

The coordinate functions $\pi_i: [0, 1]^n \rightarrow \mathbb{R}$, $(i = 1, \dots, n)$ are said to form a *free generating set* of $(M_n, 1)$ because of the following result:

Proposition 3.4. [31, 4.16] *$(M_n, 1)$ is generated by the elements π_1, \dots, π_n together with the unit 1. For every unital ℓ -group (H, v) , n -tuple (v_1, \dots, v_n) of elements in the unit interval $[0, v]$ of (H, v) , and map $\eta: \pi_i \mapsto v_i$, η can be uniquely extended to a homomorphism of $(M_n, 1)$ into (H, v) .*

Corollary 3.5. *For each $n = 1, 2, \dots$, $(M_n, 1)$ is hopfian.*

Proof. This follows from (1), by an application of Corollary 3.2 and Theorem 8.5. \square

4. OTHER CLASSES OF HOPFIAN MV-ALGEBRAS AND UNITAL ℓ -GROUPS

Germinal ideals and quotients. Following [35, Definition 4.7], for any $n = 1, 2, \dots$ and $x \in [0, 1]^n$ the ideal \mathfrak{o}_x is defined by

$$\mathfrak{o}_x = \{f \in \mathcal{M}([0, 1]^n) \mid f \text{ identically vanishes over some open neighborhood of } x\}.$$

\mathfrak{o}_x is called the *germinal ideal of $\mathcal{M}([0, 1]^n)$ at x* . Accordingly, the quotient MV-algebra $\mathcal{M}([0, 1]^n)/\mathfrak{o}_x$ is called the *germinal MV-algebra at x* , and for each $f \in \mathcal{M}([0, 1]^n)$, f/\mathfrak{o}_x is called the *germ of f (at x)*, and is denoted \check{f} whenever x is clear from the context.

If x happens to lie on the boundary of the n -cube, the neighborhoods of x are understood relative to the restriction topology.

In general, an MV-algebra in either class (i)-(iii) below need not fall under the hypotheses of Theorem 1.1. Thus the following theorem is not a special case of that result.

Theorem 4.1. *The following MV-algebras are hopfian:*

- (i) *Simple MV-algebras.*
- (ii) *Finitely generated MV-algebras with only finitely prime ideals.*
- (iii) *For any rational point w lying in the interior of the cube $[0, 1]^n$, the germinal MV-algebra $A = \mathcal{M}([0, 1]^n)/\mathfrak{o}_w$.*

Proof. (i) By Theorem 8.4(i)-(ii), the only endomorphism of a simple MV-algebra is identity.

(ii) Let σ be a surjective homomorphism of A onto itself, with the intent to prove that σ is injective. Let the unital ℓ -group (G, u) be defined by $\Gamma(G, u) = A$. Then for a unique surjective homomorphism $\tau: (G, u) \rightarrow (G, u)$ we have $\sigma = \Gamma(\tau)$. Let G_{group} denote the group reduct of (G, u) . The elementary theory of ℓ -groups and their lexicographic products [1, Theorem 3.2] shows that G_{group} is a finite product of cyclic groups \mathbb{Z} . It follows that G_{group} is hopfian ([27, §3.3 20(d) p. 152]). Since, in particular, τ is a homomorphism of G_{group} onto itself, τ is injective. Going back to (G, u) and A , also the homomorphism $\sigma = \tau \upharpoonright [0, 1]$ is injective, and A is hopfian.

(iii) Let S^{n-1} be the set of unit vectors in \mathbb{R}^n . Let $f, g \in \mathcal{M}([0, 1]^n)$. By [35, Proposition 4.8],

$$f/\mathfrak{o}_w = g/\mathfrak{o}_w \text{ iff } f(w) = g(w) \text{ and } \partial f(w)/\partial e = \partial g(w)/\partial e \text{ for all } e \in S^{n-1}. \quad (2)$$

Thus we may unambiguously speak of *the directional derivative of a germ $\check{f} \in A$* and write

$$\partial \check{f}(w)/\partial d = \partial g(w)/\partial d, \quad d \in \mathbb{R}^n,$$

independently of the actual representative $g \in \mathcal{M}([0, 1]^n)$ of \check{f} . Similarly, we may unambiguously define $\check{f}(w) = g(w)$ and say that *the germ \check{f} has value $g(w)$* .

The germinal MV-algebra A has exactly one maximal ideal, namely the set \mathfrak{g}_w given by

$$\mathfrak{g}_w = \{\check{g} \in A \mid g \in \mathcal{M}([0, 1]^n) \text{ vanishes only at } w \text{ in an open neighborhood of } w\}.$$

For every prime ideal \mathfrak{p} of A the only maximal ideal of A containing \mathfrak{p} is \mathfrak{g}_w . The ideals of A containing \mathfrak{p} are all prime, and form a finite chain (under inclusion), whose greatest element is \mathfrak{g}_w . We say that a prime ideal \mathfrak{q} of A belongs to $\text{sub } \mu(A)$ if ($\mathfrak{q} \neq \mathfrak{g}_w$ and) the only ideal properly containing \mathfrak{q} is \mathfrak{g}_w .

Claim. Equipped with the topology inherited from $\text{Spec}(A)$ by restriction, $\text{sub } \mu(A)$ is homeomorphic to the $(n - 1)$ -sphere,

$$\text{sub } \mu(A) \cong S^{n-1}. \quad (3)$$

As a matter of fact, let $\mathfrak{p} \in \text{sub } \mu(A)$ and $\check{f} \in \mathfrak{p}$. It is not hard to see that the directional derivative $\partial \check{f}(w)/\partial d$ of \check{f} at w must vanish along some direction $d \in S^{n-1}$. For otherwise, since the map $d \in S^{n-1} \mapsto \partial f(w)/\partial d \in \mathbb{R}$ is continuous and S^{n-1} is compact, there is $\rho > 0$ such that $\partial f(w)/\partial d \geq \rho$ for all $d \in S^{n-1}$. Thus every function $g \in \mathcal{M}([0, 1]^n)$ vanishing at w is dominated by a suitable integer multiple of f . It follows that \mathfrak{p} coincides with the maximal ideal \mathfrak{g}_w of A , against our hypotheses that $\mathfrak{p} \subsetneq \mathfrak{g}_w$. Another application of the continuity of the map $d \mapsto \partial \check{f}(w)/\partial d$ shows that the set $S_{\check{f}}$ of unit vectors v such that $\partial \check{f}(w)/\partial v = 0$ is a *closed* nonempty subset of S^{n-1} . A compactness argument now shows that the intersection of the sets $S_{\check{g}}$, where \check{g} ranges over all elements of \mathfrak{p} , is a closed nonempty subset $D_{\mathfrak{p}}$ of S^{n-1} . Since the point w is rational and \mathfrak{p} is prime, a routine separation argument shows that this set must be a singleton, $D_{\mathfrak{p}} = \{d_{\mathfrak{p}}\}$ with $d_{\mathfrak{p}} \in S^{n-1}$.

Conversely, every unit vector $d \in S^{n-1}$ determines the prime ideal \mathfrak{p}_d of A consisting of the germs of A vanishing at w and whose directional derivative at w along d vanishes. Since w is a rational point, \mathfrak{p}_d does not coincide with \mathfrak{g}_w . The first part of the proof of the claim can be used to show that there is no prime ideal \mathfrak{p} between \mathfrak{g}_w and \mathfrak{p}_d . So \mathfrak{p}_d belongs to $\text{sub } \mu(A)$.

In conclusion, the maps

$$\mathfrak{p} \in \text{sub } \mu(A) \mapsto d_{\mathfrak{p}} \in S^{n-1} \text{ and } d \in S^{n-1} \mapsto \mathfrak{p}_d \in \text{sub } \mu(A) \quad (4)$$

are inverses of each other. By definition of the prime spectral topology, these maps are homeomorphisms. Our claim is settled.

For every $g \in \mathcal{M}([0, 1]^n)$ we can now expand (2) as follows:

$$g \in \mathfrak{o}_w \text{ iff } \check{g}(w) = 0 \text{ and } \partial \check{g}(w)/\partial d \text{ for all } d \in \mathbb{R}^n \text{ iff } \check{g} \in \mathfrak{p} \text{ for all } \mathfrak{p} \in \text{sub } \mu(A). \quad (5)$$

To conclude the proof, by way of contradiction let $\sigma: A \rightarrow A$ be a surjective endomorphism which is not injective, i.e., there exists $h \in \mathcal{M}([0, 1]^n)$ satisfying $0 \neq \check{h} \in \ker(\sigma)$. Since, trivially, $\check{h}(w) = 0$, then by (2) we must have $\partial \check{h}(w)/\partial r > 0$ for some $r \in S^{n-1}$. Thus \check{h} does not belong to the prime ideal \mathfrak{p}_r of (4). It follows from (5) that

$$\ker(\sigma) \not\subseteq \mathfrak{p}_r \in \text{sub } \mu(A). \quad (6)$$

We now inspect the prime spectrum of the quotient MV-algebra $A/\ker(\sigma)$. The elementary ideal theory of MV-algebras, [15, pp. 16-18], yields a one-one inclusion

preserving correspondence between $\text{Spec}(A/\ker(\sigma))$ and the set of prime ideals of A containing $\ker(\sigma)$. Since this correspondence is also a homeomorphism, the set $\text{sub } \mu(A/\ker(\sigma))$ equipped with the restriction topology is homeomorphic to the set \mathfrak{P} of all $\mathfrak{p} \in \text{sub } \mu(A)$ containing $\ker(\sigma)$. Now (3) and (6) show that \mathfrak{P} is homeomorphic to a *proper* subset of S^{n-1} . As is well known, S^{n-1} is not homeomorphic to any of its proper subsets (see, e.g., [37, 7.2(6), p. 180] for a proof). As a consequence,

$$\text{sub } \mu(A/\ker(\sigma)) \not\cong \text{sub } \mu(A).$$

By contrast, from the isomorphism $A = \sigma(A) \cong A/\ker(\sigma)$ we obtain $\text{Spec}(A) \cong \text{Spec}(A/\ker(\sigma))$, $\mu(A) \cong \mu(A/\ker(\sigma))$. Since the homeomorphism of $\mu(A)$ onto $\mu(A/\ker(\sigma))$ is order preserving (and so is its inverse) we finally obtain

$$\text{sub } \mu(A) \cong \text{sub } \mu(A/\ker(\sigma)),$$

a contradiction showing that A is hopfian. \square

Corollary 4.2. *The following classes of unital ℓ -groups are hopfian:*

- (i) *Simple unital ℓ -groups.*
- (ii) *Finitely generated unital ℓ -groups with only finitely many prime ideals.*
- (iii) *For w a rational point lying in the interior of the cube $[0, 1]^n$, the germinal unital ℓ -group (G, u) at w , defined by $\Gamma(G, u) = \mathcal{M}([0, 1]^n)/\mathfrak{o}_w$. (See [6, §10.5] for an equivalent definition independent of MV-algebras).*

Proof. By Theorems 4.1 and 8.5. \square

Theorem 4.3. *Let A be a semisimple MV-algebra.*

- (i) *If $\mu(A)$ is an n -dimensional manifold without boundary then A is hopfian.*
- (ii) *If $\mu(A)$ is an n -manifold with boundary, and A has a generating set with n elements, then A is hopfian.*
- (iii) *More generally, A is hopfian if A has a generating set of n elements and $\mu(A)$ is homeomorphically embeddable onto a subset X of $[0, 1]^n$ coinciding with the closure of its interior.*

Proof. (i) Let κ be the cardinality of a generating set of A . By Theorems 8.2(iii) and 8.4, we may identify the semisimple MV-algebra A with the MV-algebra $\mathcal{M}(N)$ for some homeomorphic copy $N \subseteq [0, 1]^\kappa$ of the maximal spectral space $\mu(A)$. N is an n -manifold with boundary embedded into $[0, 1]^\kappa$.

By way of contradiction, suppose σ is a homomorphism of $\mathcal{M}(N)$ onto $\mathcal{M}(N)$ and σ is not one-one. By Lemma 8.1 we have an isomorphism

$$\mathcal{M}(N)/\ker \sigma \cong \mathcal{M}(N). \quad (7)$$

Let $N' = \bigcap \{f^{-1}(0) \mid f \in \ker(\sigma)\}$. By Theorem 8.4, $N' \cong \mu(\mathcal{M}(N'))$. Since $\ker(\sigma) \neq \{0\}$, N' is a proper nonempty closed subset of N , ([15, Proposition 3.4.2] and definition of maximal spectral topology). Since $\sigma(\mathcal{M}(N)) = \mathcal{M}(N)$ is semisimple, $\ker(\sigma)$ is an intersection of maximal ideals. By Proposition 8.3, $\mathcal{M}(N)/\ker \sigma \cong \mathcal{M}(N')$. By (7), $\mathcal{M}(N)$ and $\mathcal{M}(N')$ are isomorphic, whence their respective maximal spectral spaces are homeomorphic, and so are N' and N .

However, since N is a manifold without boundary, every one-one continuous map of N into N is surjective. (See [39, Corollary 6.3] for a proof.) We have thus reached a contradiction, showing that A is hopfian.

(ii) For some integer $n > 0$, Theorems 8.2(iii) and 8.4, enable us to identify $\mu(A)$ with a closed subset X of $[0, 1]^n$, and write $A = \mathcal{M}(X)$. Our hypothesis about $\mu(A)$ entails that X is equal to the closure of its interior in $[0, 1]^n$. Thus the rational points are dense in X , whence (by Theorem 8.4), the finite rank maximal ideals of

A are dense in $\mu(A)$. By Theorem 2.1, A is residually finite. By hypothesis, A is finitely generated. By Corollary 2.2, A is hopfian.

(iii) Same proof as for (ii). \square

Corollary 4.4. *Let (G, u) be a semisimple unital ℓ -group. Suppose the maximal spectral space of (G, u) is a manifold without boundary. Then (G, u) is hopfian.*

5. HOPFIAN VS. NON-HOPFIAN

In general, the finite rank maximal ideals of an n -generator semisimple MV-algebra A are not dense in $\mu(A)$, even when $n = 1$. So the following result is not derivable from Theorem 1.1:

Proposition 5.1. *Every one-generator semisimple MV-algebra A is hopfian.*

Proof. A is a quotient of the free one-generator MV-algebra $\mathcal{M}([0, 1])$. By Theorems 8.2(iii) and 8.4, we may identify the semisimple MV-algebra A with the MV-algebra $\mathcal{M}(X)$ for some homeomorphic copy $X \subseteq [0, 1]$ of the maximal spectral space $\mu(A)$.

By way of contradiction, suppose there is a homomorphism σ of $\mathcal{M}(X)$ onto $\mathcal{M}(X)$ which is not one-one. By Lemma 8.1 we have an isomorphism

$$\tau: \mathcal{M}(X)/\ker \sigma \cong \mathcal{M}(X) \text{ with } \tau(x/\ker(\sigma)) = \sigma(x) \text{ for all } x \in \mathcal{M}(X). \quad (8)$$

Let $X' \subseteq X$ be the intersection of the zerosets $f^{-1}(0)$ of the functions $f \in \ker(\sigma)$. By Theorem 8.4, X' is homeomorphic to $\mu(\mathcal{M}(X'))$. Since σ is not one-one, $\ker(\sigma) \neq \{0\}$. By definition of maximal spectral topology, X' is a *proper* nonempty closed subset of X , (see [15, Proposition 3.4.2] for details).

Note that $\ker(\sigma)$ coincides with the intersection of all maximal ideals containing it (because $\sigma(\mathcal{M}(X)) = \mathcal{M}(X) = A$, which is assumed to be semisimple). By [15, Proposition 3.4.5],

$$\mathcal{M}(X)/\ker \sigma \cong \mathcal{M}(X').$$

Then by (8) we have an isomorphism

$$\eta: \mathcal{M}(X) \cong \mathcal{M}(X').$$

Correspondingly, η yields a homeomorphism

$$\theta: X' \cong X \quad (9)$$

such that

$$\mathcal{M}(X')/\mathfrak{m} \cong \mathcal{M}(X)/\theta(\mathfrak{m}) \text{ for each } \mathfrak{m} \in \mu(\mathcal{M}(X')).$$

By Theorem 8.4(i), both quotients $\mathcal{M}(X')/\mathfrak{m}$ and $\mathcal{M}(X)/\theta(\mathfrak{m})$ are uniquely isomorphic to the same subalgebra $I_{\mathfrak{m}}$ of the standard MV-algebra $[0, 1]$, and we can safely write

$$\mathcal{M}(X')/\mathfrak{m} = \mathcal{M}(X)/\theta(\mathfrak{m}) = I_{\mathfrak{m}} \text{ for each } \mathfrak{m} \in \mu(\mathcal{M}(X')).$$

Thus for every $x \in X'$, θ preserves the group

$$G_x = \mathbb{Z}x + \mathbb{Z} = \mathcal{M}(X)/\mathfrak{m}_x$$

generated by x and the unit 1 in the additive group \mathbb{R} . In symbols,

$$G_x = G_{\theta(x)}. \quad (10)$$

In particular, θ preserves the denominators of rational points of X' , (if any).

Since X' is a proper subset of X , let

$$c \in X \setminus X'.$$

Case 1. c is rational. Say $\text{den}(c) = d$. There are only finitely many points in $[0, 1]$ of denominator d . By the pigeonhole principle it is impossible that θ^{-1} maps the set $D \subseteq X$ of points of denominator d one-one onto $D \setminus \{c\}$.

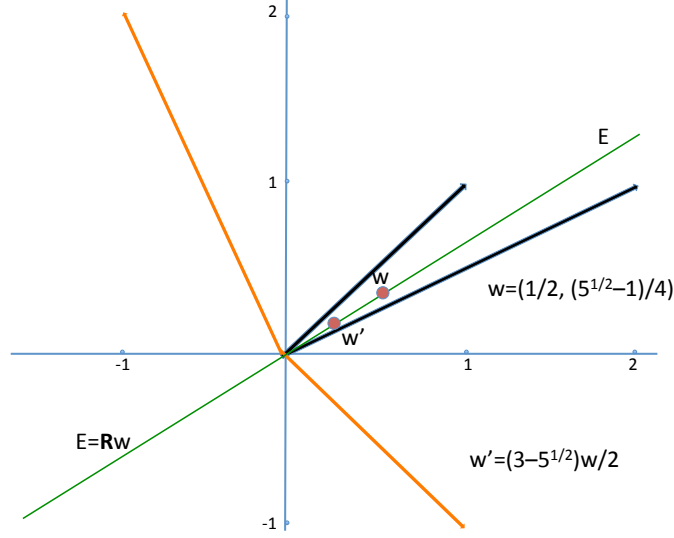


FIGURE 1. The line E is one of the two eigenspaces of the unimodular matrix with rows $(2,1)$ and $(1,1)$, and eigenvalue $1 < 1/\lambda \notin \mathbb{Q}$. E is also an eigenspace of the inverse matrix L . For each $v \in E$, $Lv = \lambda v = (3 - 5^{1/2})v/2$. Let $w = (1/2, (5^{1/2} - 1)/4) \in E \cap [0, 1/2]^2$. Then L yields a one-one \mathbb{Z} -map l^* of $[0, w]$ onto $[0, w'] = [0, Lw] \subsetneq [0, w]$. Theorem 5.2 shows that the MV-algebra $\mathcal{M}([0, w])$ is not hopfian.

Case 2. c is irrational. For any two irrational points $a, b, \in [0, 1]$ a routine verification shows $G_a = G_b$ iff $a = b$ or $a = 1 - b$. Upon writing $c \neq \theta^{-1}(c) = \theta^{-1}(\theta^{-1}(c))$ we obtain a contradiction with the fact that the one-one map θ of (9) has the preservation property (10).

Having thus proved that both cases are impossible, we conclude that A is hopfian. \square

As will be shown by our next result, when $n \geq 2$ the analog of Proposition 5.1 fails in general for n -generator semisimple MV-algebras.

Throughout the rest of this section the adjective “linear” is understood in the homogeneous sense.

Theorem 5.2. *Let L be an $n \times n$ matrix with integer entries and determinant equal to ± 1 . Suppose L has a one-dimensional linear eigenspace E with eigenvalue $0 < \lambda < 1$, and E has a nonempty intersection with the interior of $[0, 1]^n$. Then the semisimple n -generator MV-algebra $\mathcal{M}(E \cap [0, 1/2]^n)$ is not hopfian and is not residually finite.*

Proof. L acts on vectors $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ as the linear transformation $x \mapsto Lx$, also denoted L . By hypothesis, L preserves denominators of rational points.

Observe that no nonzero rational point lies in E . For otherwise (absurdum hypothesis) if $0 \neq r \in E \cap \mathbb{Q}^n$ then from $0 < \lambda < 1$ we get an infinite set $r, L(r), L(L(r)), \dots$ of rational points in $[0, 1]^n$ all of denominator $\text{den}(r)$. This is impossible, because for each $d = 1, 2, \dots$, the cube $[0, 1]^n$ contains only finitely many rational points of denominator d .

Let us write for short $E' = E \cap [0, 1/2]^n$. In view of Theorem 8.4, since E' has no nonzero rational point then a fortiori the finite rank maximal ideals of the MV-algebra $\mathcal{M}(E')$ do not form a dense set in the maximal spectral space $\mu(\mathcal{M}(E'))$. By Theorem 2.1, the semisimple MV-algebra $\mathcal{M}(E')$ is not residually finite.

Since $0 < \lambda < 1$, there is an open neighborhood \mathcal{N} of E' in $[0, 1]^n$ such that the restriction $L \upharpoonright \mathcal{N}$ maps \mathcal{N} into $[0, 1]^n$. Without loss of generality, the closure of \mathcal{N} is a rational polyhedron, whence by [35, Proposition 3.2], $L \upharpoonright \mathcal{N}$ is extendible to a \mathbb{Z} -map $l: [0, 1]^n \rightarrow [0, 1]^n$. Let the map $l^*: E' \rightarrow E'$ be defined by

$$l^* = l \upharpoonright E' = L \upharpoonright E'.$$

For any $f \in \mathcal{M}(E')$ the composite function $f \circ l^*$ is a member of $\mathcal{M}(E')$. So let the map $\sigma: \mathcal{M}(E') \rightarrow \mathcal{M}(E')$ be defined by

$$\sigma(f) = f \circ l^*, \text{ for all } f \in \mathcal{M}(E').$$

Claim. σ is a homomorphism of $\mathcal{M}(E')$ onto itself.

As a matter of fact, the inverse L^{-1} determines a linear transformation $x \mapsto L^{-1}x$ of the vector space \mathbb{R}^n one-one onto \mathbb{R}^n . Since L^{-1} is an integer matrix and the nonzero extreme of the segment E' lies in the interior of $[0, 1]^n$, for each $i = 1, \dots, n$ there is a McNaughton function $k_i: [0, 1]^n \rightarrow [0, 1]$ such that $(k_i \circ l)(x_1, \dots, x_n) = x_i$ for all $x \in E'$. The restriction $k_i^* = k_i \upharpoonright E'$ is a McNaughton function on E' satisfying the identity $(k_i^* \circ l^*)(x_1, \dots, x_n) = x_i$ for all $x \in E'$. Since each component of the identity \mathbb{Z} -map on E' is in the range of σ then, by definition of σ , every member of $\mathcal{M}(E')$ is in the range of σ , and our claim is settled.

Now, l^* is one-one but is not surjective, because it shrinks E' to the closed set $l^*(E') = \lambda E' \subsetneq E'$. There is an open rational n -simplex $T \subseteq [0, 1]^n$ such that $T \cap (E' \setminus \lambda E') \neq \emptyset$ and $T \cap \lambda E' = \emptyset$. By [35, Corollary 2.10] there is a McNaughton function $g \in \mathcal{M}([0, 1]^n)$ whose zeroset coincides with the (closed) rational polyhedron $[0, 1]^n \setminus T$. So g vanishes identically on $\lambda E'$, but does not vanish identically over E' . The restriction $g \upharpoonright E'$ is a member of $\mathcal{M}(E')$, and so is the composite function $(g \upharpoonright E') \circ l^* = \sigma(g)$. Since $(g \upharpoonright E') \circ l^*$ vanishes identically over E' , $g \upharpoonright E'$ is a nonzero member of $\ker(\sigma)$. We have just shown that the surjective homomorphism σ is not injective. Thus $\mathcal{M}(E')$ is not hopfian. \square

See Figure 1 for an example of a semisimple non-hopfian two-generator MV-algebra.

Corollary 5.3. *With the notation of the foregoing theorem, let the unital ℓ -group (G, u) be defined by $\Gamma(G, u) = \mathcal{M}(E \cap [0, 1/2]^n)$. Then (G, u) is finitely generated, semisimple and non-hopfian.*

We refer to [35, §7] for free products of MV-algebras.

The *Chang algebra* $C = \{0, \epsilon, 2\epsilon, \dots, 1-2\epsilon, 1-\epsilon, 1\}$ is defined as $\Gamma(\mathbb{Z} +_{lex} \mathbb{Z}, (1, 0))$, where $+_{lex}$ denotes the lexicographic product, [6, p. 141].

Theorem 5.4. *The following MV-algebras are not hopfian:*

- (i) *Free MV-algebras over infinitely many free generators.*
- (ii) *The free product $C \amalg C$.*
- (iii) *All countable boolean algebras.*

Proof. (i) Let $Free_\omega = \mathcal{M}([0, 1]^\omega)$ be the free MV-algebra over the free generating set $\{\pi_1, \pi_2, \dots\}$, where π_i is the i th coordinate function over the Hilbert cube $[0, 1]^\omega$. Let σ be the unique endomorphism of $Free_\omega$ extending the map $\pi_1 \mapsto \pi_1, \pi_{n+1} \mapsto \pi_n, n = 1, 2, \dots$. Then σ is a surjective non-injective homomorphism, showing that $Free_\omega$ is not hopfian. The case of $Free_\kappa$ for a cardinal $\kappa > \omega$ is similar.

(ii) For this proof we assume familiarity with MV-algebraic free products ([35, §7]) and with the basic properties of the free ℓ -group \mathcal{A}_2 over two free generators: this is the ℓ -group of all continuous piecewise *homogeneous* linear functions $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, where each piece has integer coefficients, [1, Theorem 6.3]. Directional derivatives and values of germs are understood and denoted as in the proof of Theorem 4.1(iii).

It is not hard to construct an isomorphism γ of Chang algebra C onto the germinal quotient $\mathcal{M}([0, 1])/\mathfrak{o}_0$ of $\mathcal{M}([0, 1])$ at the origin 0. To this purpose, we note that two McNaughton functions $f, g \in \mathcal{M}([0, 1])$ have the same germ at 0 iff $f(0) = g(0)$ and $\partial f(0)/\partial x^+ = \partial g(0)/\partial x^+$. So any germ \check{f} with value 0 is uniquely determined by the value $\partial \check{f}(0)/\partial x^+ \in \{0, 1, 2, \dots\}$ of its directional derivative along the positive direction of the x -axis. The map γ sends the element $n\epsilon \in C$ to the germ $n\check{x}$ whose directional derivative is n . In particular, the smallest infinitesimal ϵ of C corresponds to the germ \check{x} of the identity function x on $[0, 1]$. Symmetrically, γ sends the element $1 - m\epsilon \in C$ to the germ $\neg m\check{x}$ with value 1 and directional derivative equal to $-m$. We will henceforth identify C and $\mathcal{M}([0, 1])/\mathfrak{o}_0$.

To construct the free product MV-algebra CHC, following [35, Theorem 7.1], for every $f \in \mathfrak{o}_0$ we first embed f into $\mathcal{M}([0, 1]^2)$ by *cylindrification along the y axis*: in other words, we transform f into the function $f \circ \pi_1 \in \mathcal{M}([0, 1]^2)$, with $\pi_1: [0, 1]^2 \rightarrow [0, 1]$ the first coordinate function. We similarly transform f into the function $f \circ \pi_2$, its cylindrification along the x axis. A routine verification shows that the ideal of $\mathcal{M}([0, 1]^2)$ generated by the set of all these cylindrified functions coincides with the germinal ideal $\mathfrak{o}_{(0,0)}$ of $\mathcal{M}([0, 1]^2)$ at the origin $(0, 0) \in [0, 1]^2$. Every germ $\check{g} \in \mathcal{M}([0, 1]^2)/\mathfrak{o}_{(0,0)}$ can only have the two possible values 0 and 1. We have an isomorphism

$$\text{CHC} \cong \mathcal{M}([0, 1]^2)/\mathfrak{o}_{(0,0)}. \quad (11)$$

Let Q denote the first quadrant in \mathbb{R}^2 . A germ $\check{f} \in \mathcal{M}([0, 1]^2)/\mathfrak{o}_{(0,0)}$ having value 0 is completely characterized by the unique continuous piecewise *homogeneous* linear function $\vec{f}: Q \rightarrow \mathbb{R}_{\geq 0}$ whose directional derivatives at $(0, 0)$ along all (unit) vectors $d \in Q$ coincide with those of f . Each linear piece of \vec{f} has integer coefficients. If $\check{g} = \check{f}$ then $\vec{g} = \vec{f}$. Conversely, for any continuous piecewise homogeneous linear function $l: Q \rightarrow \mathbb{R}_{\geq 0}$ with integer coefficients there is precisely one germ $\check{h} \in \mathcal{M}([0, 1]^2)/\mathfrak{o}_{(0,0)}$ such that $l = \vec{h}$. This one-one correspondence provides a convenient identification of the set of germs in $\mathcal{M}([0, 1]^2)/\mathfrak{o}_{(0,0)}$ having value 0 at the origin, with the set P of continuous piecewise homogeneous linear functions $p \geq 0$ with integer coefficients, defined on Q . Elements of P are acted upon by the pointwise addition, truncated subtraction, and max min operations of \mathbb{R} . Similarly, each germ in $\mathcal{M}([0, 1]^2)/\mathfrak{o}_{(0,0)}$ with value 1 can be identified with $1 - p$ for a uniquely determined $p \in P$. Accordingly, we may write

$$\mathcal{M}([0, 1]^2)/\mathfrak{o}_{(0,0)} = P \cup (1 - P). \quad (12)$$

Let H be the ℓ -group of continuous piecewise homogeneous linear functions $f: Q \rightarrow \mathbb{R}$ with integer coefficients. Recalling the definition of the free ℓ -group \mathcal{A}_2 we can write

$$H = \mathcal{A}_2 \upharpoonright Q = \{f \upharpoonright Q \mid f \in \mathcal{A}_2\} \quad \text{and} \quad P = H^+ = \{p \in H \mid p \geq 0\}. \quad (13)$$

Let the unital ℓ -group (G, u) be defined as

$$(G, u) = (\mathbb{Z} +_{lex} H, 1), \quad (14)$$

where $+_{lex}$ is lexicographic product and the distinguished unit u is given by the constant function 1 over Q . For each integer m , G has a copy of H at level m . In more detail, each element of G is a function $l: Q \rightarrow \mathbb{R}$ of the form $m + f$, where $m \in \mathbb{Z}$ and $f \in H$. Elements l', l'' of G are acted upon by pointwise sum and

subtraction, and by the *lexicographic lattice order*: thus $l' \vee l''$ is the pointwise max of l' and l'' in case $l'(0) = l''(0)$; if $l'(0) > l''(0)$, then $l' \vee l'' = l'$; if $l'(0) < l''(0)$ then $l' \vee l'' = l''$. By (11)-(13) we have isomorphisms

$$\Gamma(G, u) \cong \mathcal{M}([0, 1]^2) / \mathfrak{o}_{(0,0)} \cong C \amalg C. \quad (15)$$

To conclude the proof it suffices to exhibit a surjective non-injective unital endomorphism σ of (G, u) . Let the map q be defined by

$$q: (x, y) \in Q \mapsto (x, y + x) \in Q.$$

Observe that $q \in G$. Let the homomorphism $\sigma: (G, u) \rightarrow (G, u)$ be defined by

$$\sigma: f \in (G, u) \mapsto f \circ q \in (G, u).$$

A direct inspection shows that σ is surjective: As a matter of fact, the first coordinate function x on Q is obtainable as $x \circ q$. The second coordinate function y is obtainable as $((y - x) \vee 0) \circ q$. Thus $\text{range}(\sigma) = G$. However, σ is not injective: for instance, the nonzero function $k \in G$ given by $k(x, y) = (x - y) \vee 0$ belongs to $\ker(\sigma)$. Thus (G, u) is not hopfian. From (14)-(15) and Theorem 8.5 it follows that CIIC is not hopfian.

(iii) [25, Corollary 4]. □

The unital ℓ -group $(M, 1)$ was introduced in [31, §4] as the Γ correspondent of the free countably generated MV-algebra, $(M, 1) = \Gamma(\mathcal{M}([0, 1]^\omega))$. By Theorems 5.4(i) and 8.5 we have:

Corollary 5.5. *$(M, 1)$ is not hopfian.*

In Corollary 2.2 the hopfian property of an MV-algebra A is shown to be a consequence of the following three conditions:

- (a) A is semisimple;
- (b) A is finitely generated;
- (c) The finite rank maximal ideals of A are dense in $\mu(A)$.

Our next result exhibits hopfian and non-hopfian algebras satisfying any two of these conditions, together with the negation of the third one.

Corollary 5.6. *Each of the following classes of algebras (indifferently meaning MV-algebras or unital ℓ -groups) contains a hopfian and a non-hopfian member:*

- (abc) *Non-semisimple, finitely generated algebras whose maximal ideals of finite rank are dense.*
- (a \bar{b} c) *Semisimple not finitely generated algebras, whose maximal ideals of finite rank are dense.*
- (ab \bar{c}) *Semisimple, finitely generated algebras where maximals of finite rank are not dense.*

Proof. We first deal with MV-algebras.

(\bar{a} b \bar{c}) The Chang algebra C is a hopfian example, by Theorem 4.1(ii). The free product MV-algebra CIIC is a non-hopfian example, by Theorem 5.4(ii).

(a \bar{b} c) A hopfian example is given by the uncountable atomic boolean algebra in [25, Theorem 6]. Non-hopfian examples are given by all countable boolean algebras, (Theorem 5.4(iii)).

(ab \bar{c}) A hopfian example is given by the subalgebra of $[0, 1]$ generated by $\pi/4$, (Theorem 4.1(i)). Non-hopfian examples are given by the MV-algebras considered in Theorem 5.2, notably, the two-generator MV-algebra of Figure 1.

For hopfian and non-hopfian examples of unital ℓ -groups one routinely takes the Γ correspondents of the above MV-algebras, in the light of Theorem 8.5. \square

6. APPLICATIONS TO ℓ -GROUPS

By an ℓ -group G we mean a lattice-ordered abelian group. G does not have a distinguished unit—indeed, G need not have a unit. Our main aim in this section is to show that finitely generated free ℓ -groups are hopfian and to characterize their free generating sets.

The *free n -generator ℓ -group* \mathcal{A}_n consists of all continuous piecewise homogeneous linear functions on \mathbb{R}^n with integer coefficients, [1, Theorem 6.3]. The maximal spectral space of \mathcal{A}_n is homeomorphic to the $(n-1)$ -sphere S^{n-1} .

It should be noted that the two categories of ℓ -groups and unital ℓ -groups have more differences than similarities: While ℓ -groups are definable by equations, the archimedean property of the unit in unital ℓ -groups is not even definable in first-order logic. The maximal spectral space of an ℓ -group may be empty, which is never the case for unital ℓ -groups. From the Baker-Beynon duality, [2, 4, 5] it easily follows that finitely presented ℓ -groups are dual to rational polyhedra with piecewise affine linear *rational* maps, while finitely presented unital ℓ -groups are dual to rational polyhedra with piecewise affine linear *integral* maps. This endows the latter dual pair (but not the former) with an interesting measure [35, §14], and a rich theory of “states”, [35, §10]. While the Baker-Beynon duality also implies that finitely presented and finitely generated projective ℓ -groups coincide, this is far from true of finitely generated projective unital ℓ -groups: as a matter of fact, their analysis is a tour de force in algebraic topology, [35, §17 and references therein].

This said, the hopfian property of finitely generated free ℓ -groups will be obtained from the proof that finitely generated free unital ℓ -groups and MV-algebras are hopfian.

We prepare the following lemma:

Lemma 6.1. *Every n -element generating set of the free n -generator MV-algebra $\mathcal{M}([0, 1]^n)$ is a free generating set.*

Proof. $\mathcal{M}([0, 1]^n)$ comes equipped with the free generating set $\{\pi_1, \dots, \pi_n\}$, where $\pi_i: [0, 1]^n \rightarrow [0, 1]$ is the i th coordinate function. Let $\{g_1, \dots, g_n\}$ be a generating set of $\mathcal{M}([0, 1]^n)$. Let the \mathbb{Z} -map $g: [0, 1]^n \rightarrow [0, 1]^n$ be defined by $g(x) = (g_1(x), \dots, g_n(x))$ for all $x \in [0, 1]^n$. Let $\eta: \mathcal{M}([0, 1]^n) \rightarrow \mathcal{M}([0, 1]^n)$ be the endomorphism of $\mathcal{M}([0, 1]^n)$ canonically extending the map $\pi_j \mapsto g_j$, ($j = 1, \dots, n$). Any $f \in \mathcal{M}([0, 1]^n)$ is mapped by η into the composite McNaughton function $f \circ g$. Arguing by way of contradiction, assume $\{g_1, \dots, g_n\}$ does not freely generate $\mathcal{M}([0, 1]^n)$. Then η is not an automorphism of $\mathcal{M}([0, 1]^n)$. Since η is onto $\mathcal{M}([0, 1]^n)$ then η is not one-one.

Claim. Let $R \subseteq [0, 1]^n$ denote the range of g . Observe that R is a rational polyhedron, [35, Lemma 3.4]. Let $\mathcal{M}(R)$ be the MV-algebra of restrictions to R of the McNaughton functions of $\mathcal{M}([0, 1]^n)$. Then there is an isomorphism of $\mathcal{M}([0, 1]^n)$ onto $\mathcal{M}(R)$.

Indeed, the homomorphism $\eta': \mathcal{M}(R) \rightarrow \mathcal{M}([0, 1]^n)$ defined by $h \mapsto h \circ g$ for all $h \in \mathcal{M}(R)$ is onto $\mathcal{M}([0, 1]^n)$ because the g_i generate $\mathcal{M}([0, 1]^n)$. Let $0 \neq l \in \mathcal{M}(R)$, say $l(y) \neq 0$ for some $y \in R$. There is $x \in [0, 1]^n$ with $y = g(x)$. Then $\eta'(l) = l \circ g$ does not vanish at x , which shows that η' is one-one. Our claim is proved.

Now we note that R is strictly contained in $[0, 1]^n$, for otherwise the map η' coincides with η , whence η is one-one, which is impossible. Thus there exists a point $z \in [0, 1]^n \setminus R$. Since R is a closed subset of $[0, 1]^n$ we may assume $z \in \mathbb{Q}^n$.

Let $d = \text{den}(z)$ be the least common denominator of the coordinates of z . By Theorem 8.4, the maximal ideal $\mathfrak{m}_z = \{f \in \mathcal{M}(R) \mid f(z) = 0\}$ has the property that the quotient $\mathcal{M}(R)/\mathfrak{m}_z$ is the MV-chain with $d + 1$ elements. Conversely, if $y \in R$ and $\mathcal{M}(R)/\mathfrak{m}_y$ is the MV-chain with $d + 1$ elements then $\text{den}(y) = d$. (If necessary, see [35, Proposition 4.4(iii)] for details.) For $b = 1, 2, \dots$, let $N_b(\mathcal{M}(R))$ be the number of maximal ideals $\mathfrak{m} \in \boldsymbol{\mu}(\mathcal{M}(R))$ such that $\mathcal{M}(R)/\mathfrak{m}$ has $b + 1$ elements. Then $N_b(\mathcal{M}(R))$ coincides with the (trivially finite) number of rational points s in R such that $\text{den}(s) = b$. Let $N_b(\mathcal{M}([0, 1]^n))$ be similarly defined. From $R \subsetneq [0, 1]^n$ and R being closed, it follows that $N_b(\mathcal{M}([0, 1]^n))$ is strictly larger than $N_b(\mathcal{M}(R))$ for all large b . This contradicts the existence of the isomorphism η' of $\mathcal{M}(R)$ onto $\mathcal{M}([0, 1]^n)$. \square

Remark 6.2. A different proof of Lemma 6.1 can be obtained from [24, Remark after Theorem 2] (also see [29, Corollary 8]), using the well known fact that the variety of MV-algebras is generated by its finite members, [15, proof of Chang completeness theorem 2.5.3]. However, the elementary proof given here is of some interest, because, as will be shown in the next result, it works equally well for ℓ -groups, to which [24] and [29, Corollary 8] are not apparently applicable.

Corollary 6.3. *Any generating set $\{g_1, \dots, g_n\}$ of the free n -generator ℓ -group \mathcal{A}_n is free generating.*

Proof. Let R' be the range of the map $g: x \in \mathbb{R}^n \mapsto (g_1(x), \dots, g_n(x)) \in \mathbb{R}^n$. An adaptation of the proof of the claim in Lemma 6.1, yields an isomorphism $h \mapsto h \circ g$ between \mathcal{A}_n and $\mathcal{A}_n \upharpoonright R' = \{f \upharpoonright R' \mid f \in \mathcal{A}_n\}$. Assuming by way of contradiction that $\{g_1, \dots, g_n\}$ is not free generating, we have a point $z' \in S^{n-1} \setminus R'$. We now compare the maximal spectral space S^{n-1} of \mathcal{A}_n with the maximal spectral space S' of $\mathcal{A}_n \upharpoonright R'$. The Baker-Beynon duality [2, 4, 5] shows that S' is homeomorphic to a subset of $S^{n-1} \setminus \{z'\}$. A contradiction is obtained since the $(n - 1)$ -sphere S^{n-1} is not homeomorphic to any of its proper subsets, [37, p. 180]. \square

Corollary 6.4. (i) *For all $n = 1, 2, \dots$, the free n -generator ℓ -group \mathcal{A}_n is hopfian.*

(ii) *Let the ideal \mathfrak{q} of \mathcal{A}_2 be given by all functions of \mathcal{A}_2 identically vanishing over the first quadrant Q of \mathbb{R}^2 . Then the quotient ℓ -group $\mathcal{A}_2/\mathfrak{q}$ is not hopfian.*

Proof. (i) We let $\pi_i: \mathbb{R}^n \rightarrow \mathbb{R}^n$ denote the i th coordinate function. Then \mathcal{A}_n is freely generated by the set $\{\pi_1, \dots, \pi_n\}$. Let $\omega: \mathcal{A}_n \rightarrow \mathcal{A}_n$ be a surjective homomorphism, with the intent of proving that ω is one-one. The set $E = \{\omega(\pi_1), \dots, \omega(\pi_n)\} = \{e_1, \dots, e_n\}$ generates \mathcal{A}_n . By Corollary 6.3, E is a free generating set of \mathcal{A}_n . Let the map $e: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be defined by

$$e(x) = (e_1(x), \dots, e_n(x)) \text{ for all } x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

It follows that $\text{range}(e) = \mathbb{R}^n$ and $\omega(g) = g \circ e$ for each $g \in \mathcal{A}_n$. If (absurdum hypothesis) there is a nonzero $f \in \mathcal{A}_n$ such that $0 = \omega(f) = f \circ e$, let $x \in \mathbb{R}^n$ be such that $f(x) \neq 0$. Since f identically vanishes over $\text{range}(e)$, then x does not belong to $\text{range}(e)$, a contradiction showing that ω is one-one and \mathcal{A}_n is hopfian.

(ii) As in the proof of Theorem 5.4(ii), let the ℓ -group $\mathcal{A}_2 \upharpoonright Q$ be defined by $\mathcal{A}_2 \upharpoonright Q = \{l \upharpoonright Q \mid l \in \mathcal{A}_2\}$.

Claim. Let the function $q \in \mathcal{A}_2$ be defined by $q(x, y) = 0 \vee -x \vee -y$. Then \mathfrak{q} is the ideal generated by q in \mathcal{A}_2 . In other words, for all $0 \leq l \in \mathcal{A}_2$,

$$l \in \mathfrak{q} \text{ iff } mq \geq l \text{ for some positive integer multiple } mq \text{ of } q. \quad (16)$$

As a matter of fact, if $mq \geq l$ then trivially l vanishes identically over Q , because so does q . Conversely, if $l = 0$ over Q then let $a = \partial l(1, 0)/\partial y^-$ and $b = \partial l(0, 1)/\partial x^-$. For all large integers n , $a < \partial nq(1, 0)/\partial y^-$ and $b < \partial nq(0, 1)/\partial x^-$,

whence for some suitably large $m \in \mathbb{Z}$, mq will be $\geq l$ over \mathbb{R}^2 . We have thus proved (16) and settled our claim.

As a consequence, for any two functions $f, g \in \mathcal{A}_2$ we have $f/\mathfrak{q} = g/\mathfrak{q}$ iff $|f - g|$ vanishes identically over Q iff $f \upharpoonright Q = g \upharpoonright Q$. In conclusion, we have an isomorphism $\mathcal{A}_2 \upharpoonright Q \cong \mathcal{A}_2/\mathfrak{q}$. In Theorem 5.4(ii) it is proved that $\mathcal{A}_2 \upharpoonright Q$ is a non-hopfian *unital* ℓ -group. A fortiori, $\mathcal{A}_2/\mathfrak{q}$ is a non-hopfian ℓ -group. \square

7. HOPFICITY OF THE FAREY-STERN-BROCOT AF C^* -ALGEBRA, [32], [7], [34]

Using the results of the earlier sections, in this section we will prove that the (Farey-Stern-Brocot) AF C^* -algebra \mathfrak{M}_1 has the hopfian property and is residually finite dimensional. Further, the hopfian property of \mathfrak{M}_1 extends to all its primitive quotients. Owing to its remarkable properties (see Remark 7.2), \mathfrak{M}_1 has drawn increasing attention in recent years, [7, 17, 33, 34, 36], after a latent period of over twenty years since its introduction in [32].

We refer to [16] for background on C^* -algebras, and to [18] for AF C^* -algebras, Elliott classification and K_0 . Our pace will be faster than in the previous sections.

A *unital AF C^* -algebra* \mathfrak{U} is the norm closure of an ascending sequence of finite dimensional C^* -algebras, all with the same unit. Elliott classification and further K_0 -theoretic developments [18] yield a functor

$$\mathfrak{U} \mapsto K_0(\mathfrak{U})$$

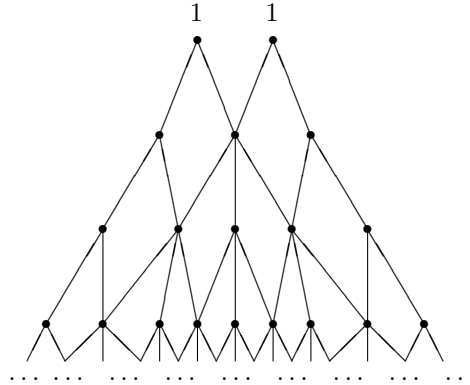
from unital AF C^* -algebras to countable unital dimension groups. K_0 is an order-theoretic refinement of Grothendieck's functor. The Murray-von Neumann order of projections in \mathfrak{U} is a lattice iff $K_0(\mathfrak{U})$ is a unital ℓ -group. K_0 sends any (always closed and two sided) ideal \mathfrak{I} of an AF C^* -algebra \mathfrak{U} to an ideal $K_0(\mathfrak{I})$ of $K_0(\mathfrak{U})$.

As the most elementary specimen of the free unital ℓ -groups introduced at the end of Section 3, $(M_1, 1)$ denotes the unital ℓ -group of all continuous piecewise (affine) linear functions $f: [0, 1] \rightarrow \mathbb{R}$, where each piece of f is a linear polynomial with integer coefficients. The constant function 1 on $[0, 1]$ is the distinguished unit of $(M_1, 1)$.

The so called Farey-Stern-Brocot unital AF C^* -algebra \mathfrak{M}_1 was originally defined in [32, §3] by

$$K_0(\mathfrak{M}_1) = (M_1, 1). \quad (17)$$

\mathfrak{M}_1 was subsequently rediscovered in [7], and denoted \mathfrak{A} . Using [32, Theorem 3.3], in [34, Theorem 1.1] the Bratteli diagram of \mathfrak{M}_1 (which is immediately obtainable from the matricial presentation [32, p. 35]), was shown to coincide with the diagram introduced in [7], namely:



The two top (depth 0) vertices are labeled 1. The label of any vertex v at depth $d = 1, 2, \dots$, is the sum of the labels of the vertices at depth $d - 1$ connected to v by an edge.

From AF C^* -algebraic K_0 -theory we have:

Corollary 7.1. *The map $\mathfrak{J} \mapsto K_0(\mathfrak{J})$ is an order isomorphism (with respect to inclusion) of the set of ideals of \mathfrak{M}_1 onto the set of ideals of $(M_1, 1)$. Further, in the category of unital dimension groups we have the isomorphism*

$$K_0\left(\frac{\mathfrak{M}_1}{\mathfrak{J}}\right) \cong \frac{K_0(\mathfrak{M}_1)}{K_0(\mathfrak{J})}, \quad (18)$$

which is automatically an isomorphism in the category of unital ℓ -groups.

Proof. The first statement follows from Elliott classification of AF C^* -algebras and subsequent K_0 -theoretic developments, [22, 1.2-1-4]. The isomorphism (18) follows because K_0 preserves short exact sequences, [19, p. 34], [18, §9]. \square

Remark 7.2. Other interesting properties of \mathfrak{M}_1 include: All primitive ideals of \mathfrak{M}_1 are essential, [34, Theorem 4.2] (also see [31, Theorem 8.4] for a precursor of this result). \mathfrak{M}_1 has a faithful invariant tracial state, [33, Theorem 3.1]. Up to isomorphism, the Effros Shen C^* -algebras \mathfrak{F}_θ are precisely the infinite dimensional simple quotients of \mathfrak{M}_1 , [32, Theorem 3.1(i)]. The center of \mathfrak{M}_1 is the C^* algebra $C[0, 1]$ of continuous complex valued functions on $[0, 1]$, [7, p. 976]. The state space of $C[0, 1]$ is affinely weak* homeomorphic to the space of tracial states on \mathfrak{M}_1 , [34, Theorem 4.5]. Every state on the center $C[0, 1]$ of \mathfrak{M}_1 has a unique tracial extension to \mathfrak{M}_1 , [17, Theorem 2.5]. In [13, Theorem 21] one finds a detailed analysis of the isomorphism classes of germinal quotients of \mathfrak{M}_1 . The automorphism group of \mathcal{M}_1 has exactly two connected components, [34, Theorem 4.3]. In [17] the Gauss map, which is a Bernoulli shift for continued fractions, is generalized in the noncommutative setting provided by \mathfrak{M}_1 .

By Theorem 8.5, the unital ℓ -group $(M_1, 1)$ inherits via Γ the semisimplicity of the free MV-algebra $\mathcal{M}([0, 1])$. Finite rank maximal ideals of $(M_1, 1)$ correspond to rational points in $[0, 1]$ via the homeomorphism of Theorem 8.4. By Theorem 2.1, semisimplicity and a dense set of finite rank maximal ideals are the counterpart for $(M_1, 1)$ of the residual finiteness of $\mathcal{M}([0, 1])$.

For the AF C^* -algebraic counterpart of the residual finiteness of the MV-algebra $\mathcal{M}([0, 1]) = \Gamma(M_1, 1) = \Gamma(K_0(\mathfrak{M}_1))$, recall that a C^* -algebra A is *residually finite-dimensional* if it has a separating family of finite dimensional representations. In other words, for any nonzero element $a \in A$ there is a finite dimensional representation π of A with $\pi(a) \neq 0$.

Theorem 7.3. *\mathfrak{M}_1 is residually finite-dimensional*

Proof. Following [31], let $\text{Maxspec}(\mathfrak{M}_1)$ denote the set of maximal ideals of \mathfrak{M}_1 with the hull kernel topology inherited from $\text{prim}(\mathfrak{M}_1)$ by restriction. (Maxspec is denoted MaxPrim in [34].) Since $(M_1, 1)$ is semisimple, the order isomorphism

$$\mathfrak{N} \in \text{Maxspec}(\mathfrak{M}_1) \mapsto K_0(\mathfrak{N}) \in \mu(M_1, 1) \quad (19)$$

of Corollary 7.1 entails

$$\bigcap \text{Maxspec}(\mathfrak{M}_1) = \{0\}. \quad (20)$$

By definition of the respective topologies of the maximal spectral spaces of \mathfrak{M}_1 and of $(M_1, 1)$, the order isomorphism (19) is also a homeomorphism

$$K_0: \text{Maxspec}(\mathfrak{M}_1) \cong \mu(M_1, 1).$$

By Theorem 8.5, the map

$$\mathfrak{n} \in \mu(M_1, 1) \mapsto \mathfrak{n} \cap [0, 1] \in \mu(\mathcal{M}([0, 1]))$$

is a homeomorphism of $\mu(M_1, 1)$ onto $\mu(\mathcal{M}([0, 1]))$. By Theorem 8.4(vi), the map

$$\xi \in [0, 1] \mapsto \mathfrak{m}_\xi \in \mu(\mathcal{M}([0, 1]))$$

is a homeomorphism of the unit real interval $[0, 1]$ onto the maximal spectral space $\mu(\mathcal{M}([0, 1]))$, (see [7, Lemma 11] or [34, (6) and Corollary 3.4] for details). From these homeomorphisms we obtain a homeomorphism

$$\rho \in [0, 1] \mapsto \mathfrak{N}_\rho \in \text{Maxspec}(\mathfrak{M}_1) \quad (21)$$

having the following property: For any rational $p/q \in [0, 1]$, the quotient $\mathfrak{M}_1/\mathfrak{N}_{p/q}$ is the C*-algebra M_q of $q \times q$ complex matrices. See [7, Proposition 4], (also see [34, Corollary 3.3(ii)]). By [32, Theorem 3.1(i)], for every irrational $\theta \in [0, 1]$ the quotient C*-algebra $\mathfrak{M}_1/\mathfrak{N}_\theta$ coincides with the Effros-Shen algebra \mathfrak{F}_θ , [18].

Let $0 \neq a \in \mathfrak{M}_1$. By (20) there is $\mathfrak{N} \in \text{Maxspec}(\mathfrak{M}_1)$ such that $a/\mathfrak{N} \neq 0$, i.e., $a \notin \mathfrak{N}$. By definition of hull-kernel topology, the set of maximal ideals \mathfrak{N} of \mathfrak{M}_1 with $a/\mathfrak{N} \neq 0$ is open in $\text{Maxspec}(\mathfrak{M}_1)$. Thus by (21) we have a rational $s/t \in [0, 1]$ such that $a/\mathfrak{N}_{s/t} \neq 0$. The quotient map $\pi: \mathfrak{M}_1 \rightarrow \mathfrak{M}_1/\mathfrak{N}_{s/t} \cong M_t$ yields the desired finite dimensional representation with $\pi(a) \neq 0$. \square

We next show that the unital AF C*-algebra \mathfrak{M}_1 inherits the hopfian property from its associated unital ℓ -group $(M_1, 1)$:

Theorem 7.4. *Suppose we are given a short exact sequence*

$$0 \rightarrow \mathfrak{I} \xrightarrow{\iota} \mathfrak{M}_1 \xrightarrow{\sigma} \mathfrak{M}_1/\mathfrak{I} \rightarrow 0$$

where \mathfrak{I} is an ideal of \mathfrak{M}_1 , ι is the identity map on $\mathfrak{I} \subseteq \mathfrak{M}_1$, and $\sigma: \mathfrak{M}_1 \rightarrow \mathfrak{M}_1/\mathfrak{I}$ is the quotient map, with $K_0(\sigma)$ a homomorphism of $K_0(\mathfrak{M}_1)$ into $K_0(\mathfrak{M}_1/\mathfrak{I})$ in the category of unital ℓ -groups. Suppose we have an isomorphism $\theta: \mathfrak{M}_1/\mathfrak{I} \cong \mathfrak{M}_1$. Then $\theta \circ \sigma$ is an automorphism of \mathfrak{M}_1 .

Proof. It is well known that \mathfrak{I} is an AF C*-algebra and $\mathfrak{M}_1/\mathfrak{I}$ is a unital AF C*-algebra, [18], [19, p. 34]. By a special case of the Bott Periodicity Theorem, [18, Corollary 9.2], the natural map

$$K_0(\sigma): K_0(\mathfrak{M}_1) \rightarrow K_0(\mathfrak{M}_1/\mathfrak{I}) \quad (22)$$

is a *surjective* homomorphism in the category of unital ℓ -groups. By Corollary 7.1, letting the ideal \mathfrak{i} of $(M_1, 1)$ be defined by $\mathfrak{i} = K_0(\mathfrak{I})$, we may rewrite (22) as follows:

$$K_0(\sigma): (M_1, 1) \rightarrow \frac{(M_1, 1)}{\mathfrak{i}}.$$

From the assumed isomorphism

$$\theta: \mathfrak{M}_1/\mathfrak{I} \cong \mathfrak{M}_1$$

the functorial properties of K_0 yield an isomorphism of unital ℓ -groups

$$K_0(\theta): K_0\left(\frac{\mathfrak{M}_1}{\mathfrak{I}}\right) \cong K_0(\mathfrak{M}_1). \quad (23)$$

Since by (18),

$$K_0\left(\frac{\mathfrak{M}_1}{\mathfrak{I}}\right) \cong \frac{K_0(\mathfrak{M}_1)}{K_0(\mathfrak{I})} = \frac{(M_1, 1)}{\mathfrak{i}},$$

then (17) and (23) yield an isomorphism

$$K_0(\theta): \frac{(M_1, 1)}{\mathfrak{i}} \cong (M_1, 1)$$

of unital dimension groups, which is automatically an isomorphism of unital ℓ -groups. By our assumption on $K_0(\sigma)$, the composite map $K_0(\theta \circ \sigma) = K_0(\theta) \circ K_0(\sigma)$ is a homomorphism of $(M_1, 1)$ onto $(M_1, 1)$ in the category of unital ℓ -groups.

By Corollary 3.5, $(M_1, 1)$ is hopfian, and hence $K_0(\theta \circ \sigma)$ is an automorphism of $(M_1, 1) = K_0(\mathfrak{M}_1)$. So both $K_0(\theta)$ and $K_0(\sigma)$ are injective, whence, by definition of K_0 , so are σ , and $\theta \circ \sigma$. It follows that $\theta \circ \sigma$ is an automorphism of \mathfrak{M}_1 . \square

Corollary 7.5. *For every primitive ideal \mathfrak{P} of \mathfrak{M}_1 , the quotient C^* -algebra $\mathfrak{M}_1/\mathfrak{P}$ inherits the hopfian property of \mathfrak{M}_1 of Theorem 7.4. This holds in particular for the Effros-Shen C^* -algebras \mathfrak{F}_θ of [18, §10] and for the Behnke-Leptin C^* -algebras $\mathcal{A}_{k,q}$ of [3, pp. 330-331].*

Proof. Both \mathfrak{F}_θ and $\mathcal{A}_{k,q}$ arise as primitive quotients of \mathfrak{M}_1 and have only a finite number of primitive quotients. The remaining primitive quotients of \mathfrak{M}_1 are finite dimensional. (See [32, Theorem 3.1(i)] and [34, Corollary 3.3] for details.) By Corollary 7.1, the unital dimension group $(G, u) = K_0(\mathfrak{M}_1/\mathfrak{P})$ is a (totally ordered) finitely generated unital ℓ -group with a finite number ($\in \{1, 2\}$) of prime ideals. By Corollary 4.2(ii), (G, u) is hopfian. Going backwards via K_0 as in the proof of Theorem 7.4, we conclude that both \mathfrak{F}_θ and $\mathcal{A}_{k,q}$ have the desired hopfian property. \square

8. APPENDIX: BACKGROUND ON MV-ALGEBRAS

Here we collect a number of basic MV-algebraic results which have been repeatedly used in the previous sections. Each result comes with a reference where the interested reader can find a proof.

8.1. Representation.

Lemma 8.1. [15, 1.2.8] *Let A and B be MV-algebras. If σ is a homomorphism of A onto B then there is an isomorphism τ of $A/\ker(\sigma)$ onto B such that $\tau(x/\ker(\sigma)) = \sigma(x)$ for all $x \in A$.*

Theorem 8.2. (i) [15, Theorem 3.5.1] *An MV-algebra A is simple iff it is isomorphic to a subalgebra of the standard MV-algebra $[0, 1]$.*

(ii) [15, Theorem 9.1.5] *For each cardinal κ , the free MV-algebra Free_κ on κ free generators is given by the McNaughton functions over $[0, 1]^\kappa$, with pointwise operations, i.e., those functions $g : [0, 1]^\kappa \rightarrow [0, 1]$ such that there are ordinals $\alpha(0) < \dots < \alpha(m-1) < \kappa$ and a McNaughton function f over $[0, 1]^m$ having the following property: for each $x \in [0, 1]^\kappa$, $g(x) = f(x_{\alpha(0)}, \dots, x_{\alpha(m-1)})$.*

(iii) [15, Theorem 3.6.7] *An MV-algebra A with κ generators is semisimple iff for some nonempty closed subset $X \subseteq [0, 1]^\kappa$, A is isomorphic to the MV-algebra $\mathcal{M}(X)$ of restrictions to X of all functions in Free_κ .*

8.2. *Yosida duality.* For any nonempty compact Hausdorff space $X \neq \emptyset$ we let $C(X)$ denote the MV-algebra of all continuous $[0, 1]$ -valued functions on X , with the pointwise operations of the MV-algebra $[0, 1]$.

An MV-subalgebra A of $C(X)$ is said to be *separating* if for any two distinct points $x, y \in X$, there is $f \in A$ such that $f(x) = 0$ and $f(y) > 0$. Following [35], for each ideal \mathfrak{i} of A we let

$$\mathcal{Z}_\mathfrak{i} = \bigcap \{f^{-1}(0) \mid f \in \mathfrak{i}\}.$$

($\mathcal{Z}_\mathfrak{i}$ is denoted $V_\mathfrak{i}$ in [15].)

Proposition 8.3. [15, Proposition 3.4.5] *Let X be a compact Hausdorff space and A be a separating subalgebra of $\text{Cont}(X)$. Then the map $f/\mathfrak{i} \mapsto f \upharpoonright \mathcal{Z}_\mathfrak{i}$ is an isomorphism of A/\mathfrak{i} onto $A \upharpoonright \mathcal{Z}_\mathfrak{i}$ if and only if \mathfrak{i} is an intersection of maximal ideals of A .*

For every MV-algebra A , we let $\text{hom}(A)$ denote the set of homomorphisms of A into the standard MV-algebra $[0, 1]$.

Theorem 8.4. (Yosida duality, [35, Theorem 4.16]) *Let A be an MV-algebra.*

(i) *For any maximal ideal \mathfrak{m} of A there is a unique pair $(\overline{\mathfrak{m}}, I_{\mathfrak{m}})$ with $I_{\mathfrak{m}}$ an MV-subalgebra of $[0, 1]$ and $\overline{\mathfrak{m}}$ an isomorphism of A/\mathfrak{m} onto $I_{\mathfrak{m}}$.*

(ii) *The map $\ker: \eta \mapsto \ker \eta$ is a one-one correspondence between $\text{hom}(A)$ and $\mu(A)$. The inverse map sends each $\mathfrak{m} \in \mu(A)$ to the homomorphism $\eta_{\mathfrak{m}}: A \rightarrow [0, 1]$ given by $a \mapsto \overline{\mathfrak{m}}(a/\mathfrak{m})$. For each $\theta \in \text{hom}(A)$ and $a \in A$, $\theta(a) = \overline{\ker \theta}(a/\ker \theta)$.*

(iii) *The map $*$: $a \in A \mapsto a^* \in [0, 1]^{\mu(A)}$ defined by $a^*(\mathfrak{m}) = \overline{\mathfrak{m}}(a/\mathfrak{m})$, is a homomorphism of A onto a separating MV-subalgebra A^* of $C(\mu(A))$. The map $a \mapsto a^*$ is an isomorphism of A onto A^* iff A is semisimple.*

(iv) *Suppose $X \neq \emptyset$ is a compact Hausdorff space and B is a separating subalgebra of $C(X)$. Then the map $\iota: x \in X \mapsto \mathfrak{m}_x = \{f \in B \mid f(x) = 0\}$ is a homeomorphism of X onto $\mu(B)$. The inverse map ι^{-1} sends each $\mathfrak{m} \in \mu(B)$ to the only element of the set $Z\mathfrak{m}$.*

(v) *From the hypotheses of (iv) it follows that $f^* \circ \iota = f$ for each $f \in B$. Thus the map $f^* \in B^* \mapsto f^* \circ \iota \in C(X)$ is the inverse of the isomorphism $*$: $B \cong B^*$ defined in (iii). In particular, $f(x) = f^*(\mathfrak{m}_x)$ for each $x \in X$.*

(vi) [35, Corollary 4.18] *For every nonempty closed subset Y of $[0, 1]^{\kappa}$, the map $\iota: x \in Y \mapsto \mathfrak{m}_x = \{f \in \mathcal{M}(Y) \mid f(x) = 0\}$ of (iv) is a homeomorphism of Y onto $\mu(\mathcal{M}(Y))$. The inverse map $\mathfrak{m} \mapsto x_{\mathfrak{m}}$ sends every maximal ideal \mathfrak{m} of $\mathcal{M}(Y)$ to the only element of $Z\mathfrak{m}$.*

Notational Stipulations. In the light of Theorem 8.4(i)-(iii), for every MV-algebra A , $a \in A$ and $\mathfrak{m} \in \mu(A)$ we will tacitly identify a/\mathfrak{m} with the real number $\overline{\mathfrak{m}}(a/\mathfrak{m})$, and write $a/\mathfrak{m} = \overline{\mathfrak{m}}(a/\mathfrak{m}) = a^*(\mathfrak{m})$. Further, if B is a separating MV-subalgebra of $C(X)$ as in Theorem 8.4(iv)-(v), identifying B with B^* and X with $\mu(B)$, we will write without fear of ambiguity, $f(x) = f(\mathfrak{m}_x) = f/\mathfrak{m}_x$ for each $x \in X$ and $f \in B$.

8.3. The Γ functor.

Theorem 8.5. (i) [31, Theorem 3.9]. *For each unital ℓ -group (G, u) let $\Gamma(G, u)$ be the MV-algebra $([0, 1], 0, \neg, \oplus)$ where $\neg x = u - x$ and $x \oplus y = \min(u, x + y)$. Further, for every homomorphism $\xi: (G, u) \rightarrow (H, v)$ let $\Gamma(\xi)$ be the restriction of ξ to $[0, u]$. Then Γ is a categorical equivalence between unital ℓ -groups and MV-algebras.*

(ii) [15, Theorem 7.2.2]. *Let $A = \Gamma(G, u)$. Then the correspondence $\phi: \mathfrak{i} \mapsto \phi(\mathfrak{i}) = \{x \in G \mid |x| \wedge u \in \mathfrak{i}\}$ is an order-isomorphism from the set of ideals of A onto the set of ideals of G , both sets being ordered by inclusion. The inverse isomorphism ψ is given by $\psi: \mathfrak{j} \mapsto \psi(\mathfrak{j}) = \mathfrak{j} \cap [0, u]$.*

(iii) [15, Theorem 7.2.4]. *For every ideal \mathfrak{j} of G , $\Gamma(G/\mathfrak{j}, u/\mathfrak{j})$ is isomorphic to the quotient MV-algebra $\Gamma(G, u)/(\mathfrak{j} \cap [0, u])$.*

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